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On the Krook–Wu Model of the Boltzmann Equation

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The distribution function of the Krook–Wu model of the nonlinear Boltzmann equation (elastic differential cross sections inversely proportional to the relative speed of the colliding particles) is obtained as a generalized Laguerre polynomial expansion where the only time dependence is provided by the coefficients. In a recent paper M. Barnsley and the present author have shown that these coefficients are recursively determined from the resolution of a nonlinear differential system. Here we explicitly show how to construct the solutions of the Krook–Wu model and study the properties of the corresponding Krook–Wu distribution functions.

KEY WORDS: Nonlinear equations; Boltzmann equation; classical physics.

1. INTRODUCTION

Recently Krook and Wu[1] have provided a model of the nonlinear Boltzmann equation of a spatially homogeneous and isotropic gas where the elastic cross section of the binary elastic collisions is inversely proportional to the relative speed. Tjon and Wu[2] made the further assumption that the generating functional of the Boltzmann normalized moments $M_n(t)$ is the Laplace transform of a new distribution function which also represents a model of the Boltzmann equation. In very recent papers Barnsley and Cornille[3] showed that the Tjon–Wu distribution function can be expanded in a series of standard Laguerre polynomials $L_{n+2}(x)$ ($x = v^2/2$, v being the velocity) in such a way that the only time dependence is provided by the coefficients $a_n(t)$. Further it was shown[3] that these $a_n(t)$, which are linear combinations of the $M_n(t)$, can be recursively obtained from the resolution of a nonlinear differential system.

In this paper we study directly the Krook–Wu distribution function $f(v, t)$ without the assumption shared by the Tjon–Wu model. If we define

$$(2\pi)^{3/2} f(v, t) = F(x = v^2/2, t)$$

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we show that F has an expansion in generalized Laguerre polynomials $L_{n+2}^{(1/2)}(x)$ with the same coefficients $a_n(t)$ as in the previous Tjon–Wu model

$$F(x, t)e^x = 1 + \sum_{n=0}^{\infty} (-1)^n a_n(t) L_{n+2}^{(1/2)}(x) \tag{1}$$

It follows that we can use the results of the previous analysis[3] for the $a_n(t)$ and replace $L_p(x)$ by $L_p^{(1/2)}(x)$ for the study of the solutions of the Krook–Wu model. In Section 2 we recall the formalism and establish the expansion (1). In Section 3 we give some sufficient conditions for the existence of $F(x, t)$ from initial conditions at $t = 0$. Taking great advantage of the generating functional of the Laguerre polynomials $L_p^{(1/2)}(x)$ [in order to start with positive distribution functions $F(x, 0)$ at $t = 0$], we study in Section 4 some classes of solutions $F(x, t)$.

2. THE FORMALISM AND THE GENERALIZED LAGUERRE EXPANSION OF $F(x, t)$

We start with the generating functional G of the normalized moments $M_n(t)$ corresponding to the Krook–Wu[1] distribution function $f(v, t)$:

$$M_n(t) = \frac{4\pi 2^n n!}{(2n+1)!} \int_0^\infty f(v, t) v^{2n+2}\, dv \tag{2}$$

$$G(\xi, t) = \sum \xi^n M_n(t) \tag{3}$$

f can be written as a transform of G

$$G(\xi, t) = 4\pi \int_0^\infty v^2 f(v, t) \left[\sum_0^\infty \frac{2^{2n} n!}{2n+1} \left(\frac{\xi v^2}{2} \right)^n \right] dv \tag{4}$$

which does not appear very simple to invert. By a change of variable

$$(2\pi)^{3/2} f(v, t) = F(x = v^2/2, t)$$

and using the identity

$$2^{2n+1} n!/(2n+1)! = \Gamma(\tfrac{1}{2})[\Gamma(n + \tfrac{3}{2})]^{-1}$$

we can rewrite Eqs. (2), (3):

$$M_n(t) = [\Gamma(n + \tfrac{3}{2})]^{-1} \int_0^\infty x^{n+1/2} F(x, t)\, dx \tag{2'}$$

$$G(\xi, t) = \int_0^\infty F(x, t) \sum_m \frac{x^{m+1/2} \xi^m}{\Gamma(m + \frac{3}{2})}\, dx \tag{4'}$$

In the following we always consider F. The $M_n(t)$ must satisfy well-defined constraints: conservation of mass and energy, and the requirement of a Maxwellian distribution at equilibrium,

$$M_0(t) \equiv M_1(t) \equiv 1, \qquad \lim_{t\to\infty} M_n(t) = 1 \tag{5}$$

As in Ref. 3, let us introduce new functions $a_n(t)$ associated with $M_n(t)$:

$$a_n(t) = \sum_{q=0}^{n+2} (-1)^q C_{n+2}^q M_{n+2-q}(t) \tag{6}$$

where $C_n{}^q$ are the usual binomial coefficients and G can be rewritten [taking into account Eq. (5)]

$$G(\xi, t) = \frac{1}{1-\xi} + \sum_{n=0}^{\infty} a_n(t)\xi^{n+2}(1-\xi)^{n+3} \tag{3'}$$

From the definition of the Γ function it is easy to see that the inverse transform of the first term of Eq. (3′) is just e^{-x}:

$$\frac{1}{1-\xi} = \sum \xi^n = \int_0^\infty e^{-x} \sum \frac{x^{n+1/2}\xi^n}{\Gamma(n+3/2)}\,dx, \qquad |\xi| < 1$$

This suggests the form

$$F = e^{-x} + \sum_n l_{n+2}(x)a_n(t)$$

where the l_n have to be determined and must satisfy

$$\frac{\xi^{n+2}}{(1-\xi)^{n+3}} \equiv \int_0^\infty l_{n+2}(x) \sum_m \frac{x^{m+1/2}\xi^m}{\Gamma(m+\frac{3}{2})}\,dx = \sum \xi^m C_m^{n+2}, \qquad |\xi| < 1$$

Finally our problem for l_n is reduced to finding the inverse Mellin transform

$$\int_0^\infty dx\,\frac{l_n(x)x^{m+1/2}}{\Gamma(m+3/2)} = C_m{}^n$$

for integer m values. We obtain[4]

$$\int_0^\infty e^{-x}\frac{L_n^{(1/2)}(x)x^{m+1/2}}{\Gamma(m+3/2)} = \begin{cases} C_m{}^n(-1)^n & \text{if} \quad n \leqslant m \\ 0 & \text{if} \quad n > m \end{cases}$$

so that F has an expansion with generalized Laguerre polynomials

$$e^x F(x, t) = 1 + \sum_{n=0}^{\infty} (-1)^n L_{n+2}^{(1/2)}(x)a_n(t)$$
$$L_0^{(1/2)} = 1 \tag{1a}$$
$$L_p^{(1/2)}(x) = \sum_{m=0}^{p-1} (-x)^m \frac{(m+\frac{3}{2})(m+\frac{5}{2})\cdots(p-\frac{1}{2})(p+\frac{1}{2})}{m!\,(p-m)!} + \frac{(-x)^p}{p!}$$

In (1a) the coefficient of $L_0^{(1/2)}$ is 1, due to the constraint $M_0 \equiv 1$, and the coefficient of $L_1^{(1/2)}$ is zero, due to $M_0(t) \equiv M_1(t) \equiv 1$. From the explicit expression of the $L_n^{(1/2)}(x)$ in powers of x we can rewrite $e^x F(x, t)$ as a power series (assuming that we can invert the order of summations)

$$\begin{aligned} e^x F(x,t) &= 1 + H_0(t) + \sum_{q=1}^{\infty} \frac{(-x)^q}{(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})\cdots(q+\frac{1}{2})} H_q(t) \\ H_0(t) &= \sum (-1)^n a_n(t)\lambda_n \\ H_q(t) &= \frac{1}{q!}\sum (-1)^n \lambda_n a_n(t)(n+2)(n+1)\cdots(n+3-q) \\ \lambda_n &= (2n+5)!!\,2^{-n-2}[(n+2)!]^{-1} \end{aligned} \tag{1b}$$

which can be justified only after a study of the properties of the $\{a_n\}$ or of the $\{H_q\}$. Sufficient conditions for absolute convergence and $t \in [0, \infty]$ will be established in the next section.

From Eq. (1) we see that the nontrivial dependence of the Krook–Wu distribution functions is entirely provided by set $\{a_n(t)\}$ and we recall briefly the method given in Ref. 3 in order to construct these $\{a_n(t)\}$ from initial values $\{a_n(0)\}$. Krook and Wu[(1)] have established that G satisfies a nonlinear partial differential equation (nl pde)

$$\left(\frac{\partial^2}{\partial\xi\,\partial t} + \frac{\partial}{\partial\xi}\right)(\xi G) = G^2 \tag{7}$$

Let us define a new variable u such that $\xi(1 + u^{-1}) = 1$, and a new generating functional $H(u, t)$ of the moments $M_n(t)$ or $a_n(t)$

$$H(u,t) = 1 + (1-\xi)G(\xi,t) = \sum_{n=0}^{\infty} a_n(t)u^{n+2} \tag{8}$$

Due to Eq. (7), H also satisfies a nl pde, which can be written for the expansion (8)

$$\sum u^{2+n}\left[(3+n)\frac{d}{dt}a_n(t) + (n+1)a_n(t)\right] \equiv \left[\sum u^{n+2}a_n(t)\right]^2 \tag{9a}$$

Equating to zero the coefficients of u^{n+2}, we get a nonlinear differential system for the a_n

$$(3+n)\frac{d}{dt}a_n(t) + (n+1)a_n(t) \equiv \sum_{m+p=n-2} a_m(t)a_p(t) \tag{9b}$$

which can be solved recursively

$$a_0(t) = a_0(0)\exp(-t/3), \qquad a_1(t) = a_1(0)\exp(-t/2)$$

$$a_n(t) = \exp\left(-\frac{n+1}{n+3}t\right)\left[a_n(0) + \int_0^t \exp\left(\frac{n+1}{n+3}t'\right) \times \sum_{m+p=n-2} a_m(t')a_p(t')(n+3)^{-1}\,dt'\right] \tag{9c}$$

The set of values $\{a_n(0)\}$ correspond to arbitrary integration constants and lead to the different possible solutions $\{a_n(t)\}$. It was shown that $a_n(t)$ decreases at least like $\exp\{-[(n+1)/(n+3)]t\}$ and consequently $\lim_{t\to\infty} a_n(t) = 0$. It follows from (1a) that at fixed x, $\lim_{t\to\infty} F(x, t) = e^{-x}$, which represents a Maxwellian behavior. Finally we recall that the $\{M_n(t)\}$ can be reconstructed from the $a_n(t)$

$$M_n(t) = 1 + \sum_{q=0}^{n-2} C_n{}^q a_{n-2-q}(t), \qquad n \geqslant 2 \tag{10}$$

3. EXISTENCE OF THE DISTRIBUTION FUNCTION $e^x F(x, t)$ EXPRESSED EITHER WITH THE POWER SERIES (1b) OR WITH THE GENERALIZED LAGUERRE SERIES (1a)

As in Ref. 3 for the Tjon–Wu model, we can, for the study of the existence of the Krook–Wu distribution function, consider two different approaches. The methods for the two models being essentially the same, with the only technical change that the $L_n^{(0)}$ must be replaced by the $L_n^{(1/2)}$, we sketch very briefly the salient results.

In the first approach, taking into account the orthogonal properties of the generalized Laguerre polynomials (with well-defined weight functions), we construct a Hilbert space of functions expanded as in (1a) in the Laguerre polynomial basis and try to find sufficient conditions in order that the solution stays in the space at later time if it is present at $t = 0$. We have to define an inner product and the key property is to find that the solutions are square-integrable in a well-defined way. Let us call $\mathscr{H}$ the Hilbert space of real-valued functions $0 \leqslant x < \infty$ defined by the symmetric inner product

$$(f, g) = \int_0^\infty f(x)g(x)w(x)\,dx, \qquad f, g \in \mathscr{H}$$

where $w(x) = e^{-x}x^{1/2}$ is the weight function of the generalized $L_n^{(1/2)}$ Laguerre polynomials. Taking into account the orthogonal property

$$\int_0^\infty w(x)L_m^{(1/2)}(x)L_p^{(1/2)}(x)\,dx = \delta_{mp}\Gamma(p+\tfrac{3}{2})/p!$$

we could introduce a complete orthonormal basis for $\mathscr{H}$ with the functions $L_n^{(1/2)}(x)[\Gamma(m+\frac{3}{2})/m!]^{-1/2}$. We assume that $F(x,0)e^x \in \mathscr{H}$ can be expanded in this basis. Further, we assume that the moments $M_0 \equiv M_1 \equiv 1$ are satisfied. It follows that $F(x,0)e^x$ has an expression like (1a) with no component corresponding to $L_1^{(1/2)}$. We define $F(x,t)e^x$, where the components at $t \neq 0$ are given by the solutions (9c) of the system (9b), and which is expanded in the same basis (1a). We want to find conditions at $t=0$ such that the norm of Fe^x at $t \neq 0$ remains finite. If we define

$$\int_0^\infty w(x)F^2(x,t)e^{2x}\,dx = \Gamma(\tfrac{3}{2})[1+\tilde{N}(t)], \qquad \tilde{N}(t) = \sum a_n{}^2(t)\lambda_n \tag{11}$$

with λ_n defined in (1b), then we must find sufficient conditions on $\tilde{N}(0)$ such that $\tilde{N}(t) < \infty$ for $t \in [0,\infty]$.

We notice that, as in Ref. 3, a trivial generalization of this study could be done. We could start with $e^xF(x,0)$, violating the moments conditions $M_0 \equiv M_1 \equiv 1$, determine the corresponding $a_n(t)$ ($n \geqslant -2$) solutions of a system generalizing Eqs. (9a)–(9c), and obtain the conditions on $\tilde{N}(0)$ [$n \geqslant -2$ in Eq. (11)] ensuring the boundedness of the corresponding $\tilde{N}(t)$ for $t \neq 0$.

In the second approach we consider the power series given by (1b) and try to obtain sufficient conditions at $t=0$ ensuring both the convergence of the series at any $t \geqslant 0$ and the existence of sums $F(x,t)$ which are entire functions in the x plane for all $t \geqslant 0$. With the modulus $|a_n(t)|$ let us define

$$\begin{aligned} N_0(t) &= \sum \lambda_n |a_n| \\ N_q(t) &= \sum \lambda_n |a_n|(n+2)(n+1)\cdots(n+3-q), \qquad q \geqslant 1 \end{aligned} \tag{12a}$$

We get absolute upper bounds for the power series (1b),

$$|e^xF(x,t)| < 1 + N_0(t) + \sum_{q=1} \frac{|x|^q\Gamma(\frac{3}{2})}{q!\,\Gamma(q+\frac{3}{2})} N_q(t) \tag{12b}$$

$N_q(t)[\Gamma(q+\frac{3}{2})]^{-1}$ is essentially an upper bound for the qth derivative with respect to x of $e^xF(x,t)$ at $x=0$. We want to obtain sufficient conditions at $t=0$ such that the set $\{N_q(t)\}$ leads to entire x functions for the lhs of Eq. (1b). Let us assume, for instance, that we have found conditions on the set $\{N_q(0)\}$ such that $N_q(t) < q! \times (\text{const})^q$, where the constant is t independent. In such a case we have absolute convergence for the sum (1b) [the inversion of the summation in Eqs. (1a) and (1b) is justified] and the rhs of Eq. (12b) is less than const $\times$ exp(const$|x|$) for any $t \geqslant 0$ [using inequalities for the $\Gamma(q + \text{const})$ functions].

Due to the possibility of different classes of solutions (9b), (9c) [arbitrariness on the set $a_n(0)$], the smallest n value for which $a_n(t) \not\equiv 0$ is not necessarily $n=0$. It can be any integer value n_0 [$a_{n_0}(t) \not\equiv 0$], $n_0 = 0, 1, 2, \ldots$. We shall find that the bounds on $\tilde{N}$, N_0, N_q depend explicitly on n_0. The study is done in the Appendix.

3.1. Bounds on $\tilde{N}(t) = \Sigma_{n=n_0}^{\infty} a_n^2(t)\lambda_n$ from Conditions on $\tilde{N}(0)$

We start from Eq. (9b), notice that the first nonlinear contribution on the rhs appears for $n = 2n_0 + 2$, multiply by $\lambda_n a_n(t)$, integrate from 0 to t, use both the Schwarz inequality and the inequality

$$\frac{\lambda_n}{\lambda_m \lambda_{n-m-2}} \leqslant \Lambda(n) = \frac{2}{15} \frac{(2n+3)(2n+5)}{(n+1)(n+2)}, \qquad 0 \leqslant m \leqslant n-2 \tag{13}$$

proved in Appendix A1, and finally sum over n to deduce, in Appendix A2, an integral inequality [see Eq. (A3)] which can be solved. If

$$\tilde{N}(0) \leqslant \left(\frac{n_0+1}{n_0+3}\right)^2 \left[\sum_{n=2n_0+2}^{\infty} \frac{\Lambda(n)}{(n+3)^2}\right]^{-1} \tag{14}$$

we obtain an explicit upper bound for $\tilde{N}(t)$ [see (A4)] such that $\tilde{N}(t) \leqslant \tilde{N}(0)$ for $t \in [0, \infty]$. Thus if $n_0 = 0, 1, 2,\ldots$, we must have $\tilde{N}(0) < 0.82, 2.82, 5.46,\ldots$. Further, if the inequality in Eq. (14) is strict, then $\lim_{t\to\infty} \tilde{N}(t) = 0$.

3.2. Bounds on $N_0(t) = \Sigma_{n=n_0}^{\infty} \lambda_n |a_n(t)|$ from Conditions on $N(0)$

We start from Eq. (9c), multiply by λ_n, take the modulus of both sides, bound the rhs by the sum of the modulus of the different terms, use both the Schwarz inequality and Eq. (13), sum over n, and deduce, in Appendix A3, an integral inequality [see Eq. (A6)] which can be solved. If

$$N_0(0) \leqslant \frac{n_0+1}{n_0+3} \frac{2n_0+5}{\Lambda(2n_0+2)} \tag{15}$$

we obtain an explicit upper bound for $N_0(t)$ [see (A7)] such that $N_0(t) \leqslant N_0(0)$ for $t \in [0, \infty]$. Thus if $n_0 = 0, 1, 2,\ldots$, we must have $N_0(0) \leqslant 2.38, 5.50, 8.89,\ldots$. Further, if the inequality in Eq. (15) is strict, then $\lim_{t\to\infty} N_0(t) = 0$.

3.3. Bounds on $N_q(t)$, $q \geqslant 1$

We start with Eq. (9a), which we differentiate $q-1$ times with respect to u, perform some algebraic manipulations, and obtain, in Appendix A4, a set of inequalities [see (A9)]

$$\begin{aligned} N_q(t) &< e^{-t}[N_q(0) + qN_{q-1}(0)] + qN_{q-1}(t) \\ &\quad + e^{-t} \int_0^t e^{t'} \left[2N_{q-1}(t') + \sum_{p=0}^{q-1} C_{q-1}^p N_p(t') N_{q-1-p}(t')\right] \end{aligned} \tag{16}$$

from which we can recursively obtain bounds on $N_q(t)$ from the set $\{N_p(0)\}, p \leqslant q$, and $N_0(t)$. We assume that the sufficient condition in Eq. (13) ensuring $N_0(t) \leqslant N_0(0)$ is satisfied.

The same set of inequalities (16) was also obtained and studied in Ref. 3, so that we quote the result which can be deduced from it: If $N_0(t) \leqslant N_0(0)$ and if

$$|N_q(0) \leqslant q!\, N_0(0)[4 + N_0(0)]^{q-1}[N_0(0) + 2/q] \qquad \text{for } q \geqslant 1$$

then from (16) it follows that

$$N_q(t) \leqslant q!\, N_0(0)[4 + N_0(0)]^q$$

3.4. Results Concerning the Existence of $\tilde{N}(t)$ in the Case of a Violation of Mass and Energy Conservation Laws

We do not require $M_0(t) \equiv M_1(t) \equiv 1$; define $a_{-2}(t) = M_0(t)$ and $a_{-1}(t) = M_1(t) - M_0(t)$, and instead of Eq. (9b) we have a new nonlinear differential system,[3]

$$(3 + n)\left(\frac{d}{dt} a_n(t) + a_n\right) = \sum_{m+q=n-2}^{\infty} a_m a_q, \qquad n = -2, -1, 0, 1, 2, \ldots \quad (9b')$$

$e^x F$ is still given by a generalized Laguerre polynomials expansion,

$$e^x F(x, t) = \sum_{n=-2}^{\infty} (-1)^n L_{n+2}^{(1/2)}(x) a_n(t)$$

and for the existence of the solution in the Hilbert space we must investigate

$$\int w F^2 e^{2x}\, dx = \Gamma(\tfrac{3}{2}) \tilde{N}(t), \qquad \tilde{N}(t) = \sum_{n=-2}^{\infty} a_n^{\,2}(t) \lambda_n \qquad (11')$$

The study is done in Appendix A5. It is shown that $\tilde{N}(t)$ is bounded for $t > 0$ (or even $t \to \infty$) if either $\tilde{N}(0) \leqslant 6\pi^{-2}$ or $\sum_{n=-2}^{\infty} \lambda_n^{1/2} |a_n(0)| \leqslant 1$.

4. CONSTRUCTION OF THE SOLUTIONS OF THE KROOK–WU MODEL

We consider initial value conditions at $t = 0$ on the set $\{a_n(0)\}$ such that $F(x, 0) > 0$. If the number of $a_n(0) \not\equiv 0$ is finite, we easily control the positivity of the sum $1 + \sum a_n(0) L_{n+2}^{(1/2)}(x)$; however, the problem becomes difficult when the set $\{a_n(0)\}$ has an infinite number of elements. Fortunately we can take great advantage of the generating functional of the Laguerre polynomials and by differentiation, integration, linear combination,..., obtain a large class of $\{a_n(0)\}$ such that the sum (1a) can be written in closed form and the positivity of $F(x, 0)$ easily established. The determination of the $\{a_n(t)\}$ being the same for either the Krook–Wu or the Tjon–Wu model, we sketch also some results of Ref. 3.

4.1. The Fundamental Solutions Where Only $a_{p-1}(0) \neq 0$, p Integer $\geqslant 1$

We recall[3] that the only $a_n(t) \not\equiv 0$ are for $n = P - 1 + k(P + 1)$, $k = 0, 1, 2, \ldots$. We define $\lambda(k) = (k + 1)(P + 1)$ and $a_{n=-2+\lambda(k)}(t) = c_k(t)$, and substituting into (9c), we get

$$c_0(t) = a_{p-1} \exp\left(-t \frac{p}{p+2}\right)$$

$$c_k(t) = \frac{1}{1 + \lambda(k)} \exp\left(-\frac{\lambda(k) - 1}{\lambda(k) + 1} t\right) \times \int_0^t \exp\left(\frac{\lambda(k) - 1}{\lambda(k) + 1} t'\right) \sum_{m+m'=k-1} c_m(t') c_{m'}(t')\, dt'$$

and the $c_k(t)$ can be obtained recursively from $c_0(t)$. The Laguerre expansion (1a) becomes

$$e^x F(x, t) = 1 + \sum_k c_k(t)(-1)^{\lambda(k)} L_{\lambda(k)}^{(1/2)}(x)$$

$$e^x F(x, 0) = 1 + a_{p-1}(0)(-1)^{P-1} L_{p+1}^{(1/2)}(x)$$

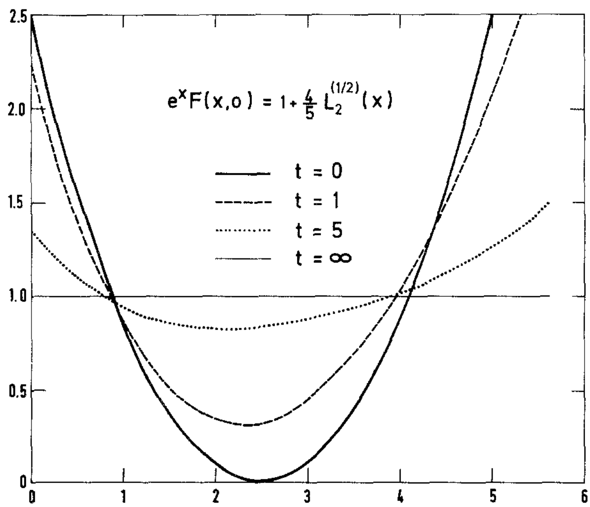

Fig. 1

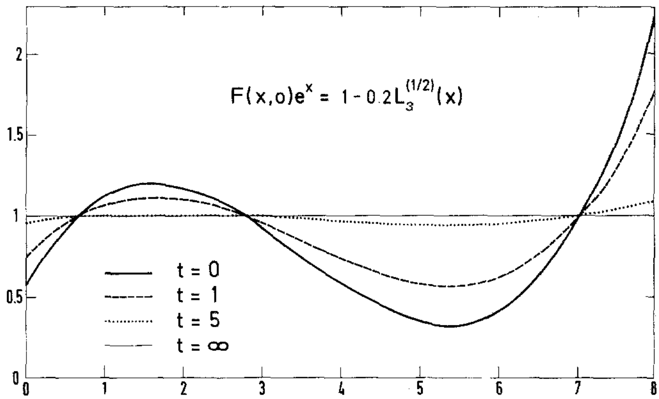

Fig. 2

and so whereas at $t = 0$ we have only one $L_n^{(1/2)}(x)$, as soon as t becomes positive, we have an infinite number of $L_n^{(1/2)}(x)$. As illustration we consider two simple examples.

In Figs. 1 and 2 we plot the ratios $\bar{F}(x, t) = F(x, t)/F(x, \infty)$ corresponding to the initial conditions $e^x F(x, 0) = 1 + \frac{4}{5} L_2^{(1/2)}(x)$ and $1 - 0.2 L_3^{(1/2)}(x)$. The results are similar to those of the Tjon–Wu model.[3] In Fig. 1 we see a small x interval (near the second zero of $\bar{F} - 1$) for not too large t values where $\bar{F}(x, t)$ is slightly larger than $\bar{F}(x, 0)$ or $\bar{F}(x, \infty)$.

4.2. Infinite Mixing of the Fundamental Solutions

Where the set $\{a_n(0)\}$ has an infinite number of elements. We start with the generating functional of the generalized Laguerre polynomials

$$1 + \sum_1^\infty z^n L_n^{(1/2)}(x) = (1 - z)^{-3/2} \exp \frac{xz}{z - 1}, \qquad |z| < 1 \tag{17}$$

Our aim is to deduce from Eq. (17) simple examples of $e^x F(x, 0)$ in closed form in such a way that the positivity of the sum of the Laguerre polynomials is easily established. We must have the coefficients of $L_0^{(1/2)}(x)$ and $L_1^{(1/2)}(x)$ respectively equal to one and zero in order to satisfy $M_0 \equiv M_1 \equiv 1$.

(i) We consider a linear combination of Eq. (17) and of its first derivative with respect to z:

$$\begin{aligned} e^x F(x, 0) &= 1 + \sum_2^\infty z^m L_m^{(1/2)}(x)(1 - m) \\ &= \frac{1}{(1 - z)^{5/2}} \left(1 - \frac{5z}{2} + \frac{xz}{1 - z}\right) \exp \frac{xz}{z - 1}, \qquad 0 < z < 1 \end{aligned} \tag{18}$$

For this example $a_n(0) = (-1)^{n+1}(n+1)z^{n+2}$; if we substitute into Eq. (9c) we find

$$a_n(t) = (-1)^{n+1}(ze^{-t/6})^{n+2}(n+1)$$

and it follows that $e^x F(x, t)$ is obtained from Eq. (18) by the substitution $z \to ze^{-t/6}$. This example corresponds to the particular Krook–Wu solution[1] or equivalently to the Bobylev[5] one, which was also written with a Laguerre expansion.

(ii) We can obtain a more general family by linear combinations of Eq. (17) with higher order derivatives and obtain for the sums exponentials multiplied by polynomials of arbitrary order in x. Let us first notice that any z derivative of Eq. (17) has a sum written down in terms of Laguerre polynomials of argument $x/(1-z)$:

$$\sum_n C_n{}^q z^n L_n^{(1/2)}(x) = (1-z)^{-3/2}\left(\exp \frac{xz}{z-1}\right)\left(\frac{z}{1-z}\right)^q L_q^{(1/2)}\left(\frac{x}{1-z}\right)$$

We add easily an arbitrary number of such terms:

$$\begin{aligned} e^x F(x, 0) &= \sum_{n=0}^{\infty} L_n^{(1/2)}(x) z^n \left(\sum_{p=0}^{q} d_p C_n{}^p\right) \\ &= (1-z)^{-3/2}\left(\exp \frac{xz}{z-1}\right) \sum_{p=0}^{q} d_p \left(\frac{z}{1-z}\right)^p L_p^{(1/2)}\left(\frac{x}{1-z}\right) \qquad (19) \end{aligned}$$

where $d_0 = 1$, $d_1 = -1$, and the other d_p are arbitrary. If the $\{a_n(0)\}$ corresponding to the lhs of Eq. (19) are given as input into Eq. (9c), then by calculating the $a_n(t)$ and substituting into (1a) we build a family of solutions where the initial conditions correspond to exponentials multiplied by arbitrary polynomials of arbitrary order q. Of course other families of solutions can be deduced from Eq. (17).

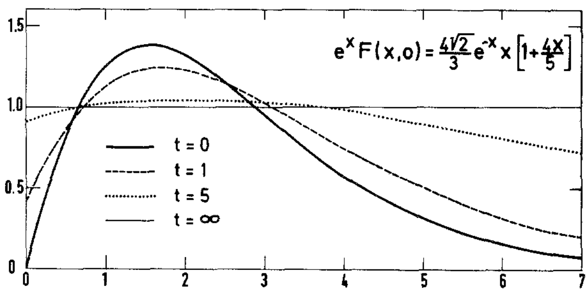

Fig. 3

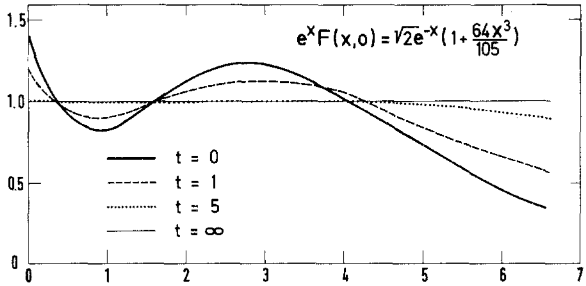

Fig. 4

(iii) Another simple family can be obtained from linear combination of Eq. (17) for different z values,

$$1 - z_1 z_2 \sum_{2} L_m^{(1/2)}(x) \sum_{p=0}^{n-2} z_1{}^p z_2^{n-p-2}$$

$$= (z_2 - z_1)^{-1}\left[\frac{z_2}{(1-z_1)^{3/2}} \exp \frac{x z_1}{z_1 - 1} - \frac{z_1}{(1-z_2)^{3/2}} \exp \frac{x z_2}{z_2 - 1}\right] \qquad (20)$$

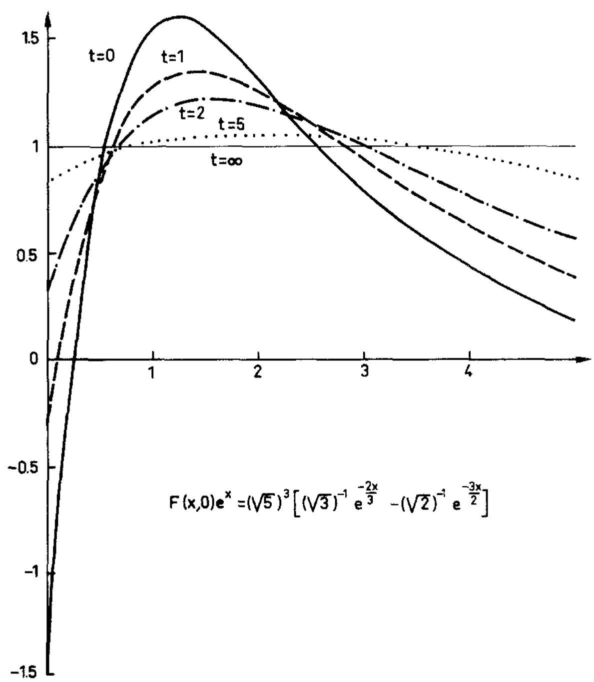

Fig. 5

In these examples Eqs. (19)–(20) represent $e^x F(x, 0)$, so that the arbitrary parameters $z, d_p, \ldots$ or z_1, z_2 are restricted in such a way that the rhs of these equations represent positive functions for $x \geqslant 0$, although there also exist examples violating positivity at $t = 0$ and not for $t > t_0 > 0$.

In Fig. 3 we plot $\bar{F}(x, t)$ corresponding to $z = \frac{1}{2}, d_2 = \frac{4}{15}, d_p = 0$ for $p > 2$ in Eq. (19), and in Fig. 4, $z = \frac{1}{2}, d_2 = \frac{4}{5}, d_3 = -\frac{8}{35}, d_p = 0$ for $p > 3$. In both cases near the largest $\bar{F} - 1$ zero we remark the same small effect as the one discussed previously in Fig. 1. In Fig. 5 we plot Eq. (20), $z_1 = \frac{2}{5}, z_2 = \frac{3}{5}$, having at $t = 0$ a negative part which disappears for $1 < t_0 < 2$ and higher t values.

5. CONCLUSION

In this paper[6] we have studied the solutions of the Krook–Wu[1] model and found that their features are very similar to those of the Tjon–Wu[2] one studied in Ref. 3, the only difference being that the expansions are with Laguerre polynomials $L_n^{(1/2)}$ instead of $L_n^{(0)}$. Since in both models the time dependence is provided by the same set $a_n(t)$, it follows that the comments concerning the Krook–Wu conjecture[1] [general structure suggested by the particular Krook–Wu solution (18) of the present paper] are the same in both cases. If M_2 and M_3 do not satisfy very particular relations, then the slowest dependences of the $a_n(t)$ are $\exp(-t/3)$ and $\exp(-t/2)$ and these time dependences are already present (in a given combination) in the particular Krook–Wu solution.

Finally we notice two possible extensions of our work, giving up the present assumptions of isotropic collisions and a spatially homogeneous gas; this has already been done for the Krook–Wu or Bobylev particular solution.[1,5,7]

APPENDIX A

A1. Bound on $\lambda_n/\lambda_m\lambda_{n-m-2}$, $m \in [0, n-2]$, $n \geqslant 2$. We recall

$$\lambda_n = (2n+5)!!/2^{n+2}(n+2)!$$

so that

$$\lambda_n/\lambda_m\lambda_{n-m-2} = (2n+5)!!/[(n+2)!\,\beta_n{}^m]$$

with

$$\beta_n{}^m = \frac{(2m+5)!!\,[2(n-m)+1]!!}{(m+2)!\,(n-m)!} = \beta_n^{n-m-2}$$

$$m \in [0, n-2], \qquad n \text{ and } m \text{ integers}, \qquad n > 2$$

However, for n fixed and $p < (n-3)/2$, $\beta_n{}^p$ is increasing:

$$\beta_n{}^p - \beta_n^{p+1} = \frac{(2p+5)!!\,[2(n-p)-1]!!}{(p+3)!\,(n-p)!}(2p+3-n)$$

It follows that $\beta_n{}^m \geqslant \beta_n{}^0$ and finally

$$\frac{\lambda_n}{\lambda_m\lambda_{n-m-2}} \leqslant \frac{\lambda_n}{\lambda_0\lambda_{n-2}} = \Lambda(n) = \frac{(2n+3)(2n+5)}{(n+2)(n+1)}\frac{2}{15} \tag{A1}$$

and we remark that $\Lambda(n)$ is a decreasing function of n.

A2. Bound on $\tilde{N}(t) = \sum_{n=n_0} a_n{}^2(t)\lambda_n$. We multiply Eq. (9b) by $\lambda_n a_n$, put $\gamma_n = (n+1)/(n+3)$, and integrate from 0 to t:

$$\begin{aligned} a_n{}^2(t)\lambda_n &= [\exp(-2\gamma_n t)]a_n{}^2(0)\lambda_n, \qquad n \leqslant 2n_0+1 \\ a_n{}^2(t)\lambda_n &= [\exp(-2\gamma_n t)]\Big\{a_n{}^2(0)\lambda_n + \frac{2}{n+3}\int_0^t [\exp(2\gamma_n t')]a_n(t')\lambda_n^{1/2} \\ &\quad \times \sum_{m+p=n-2} \lambda_n^{1/2} a_m(t')a_p(t')\,dt'\Big\}, \qquad n \geqslant 2(n_0+1) \end{aligned} \tag{A2}$$

From the Schwarz inequality and (A1) we obtain

$$\Big|\sum_{m+p=n-2} \lambda_n^{1/2} a_m a_p\Big| \leqslant \Lambda^{1/2}(n) \sum |\lambda_m^{1/2} a_m||\lambda_p^{1/2} a_p| < \Lambda^{1/2}(n)\tilde{N}$$

We remark that $\exp(-\gamma t)\int_0^t (\exp \gamma t')|f(t')|\,dt'$ is a decreasing function of γ and from (A2) we get

$$\begin{aligned} a_n{}^2(t)\lambda_n &\leqslant [\exp(-2\gamma_{n_0} t)]a_n{}^2(0)\lambda_n \\ a_n{}^2(t)\lambda_n &< [\exp(-2\gamma_{n_0} t)]\Big\{a_n{}^2(0)\lambda_n + \frac{2}{n+3}\Lambda^{1/2}(n) \\ &\quad \times \int_0^t [\exp(2\gamma_{n_0} t')]|a_n(t')\lambda_n^{1/2}|\tilde{N}(t')\,dt'\Big\} \end{aligned}$$

Summing over n and using the Schwarz inequality for $\sum |a_n|\Lambda^{1/2}(n+3)^{-1}\lambda_n$, we get a nonlinear integral inequality:

$$\begin{aligned} \tilde{N}(t)\exp(2\gamma_{n_0} t) &\leqslant \tilde{M}(t) = \tilde{N}(0) + 2C_{n_0}\int_0^t [\exp(2\gamma_{n_0}^{t'})]\tilde{N}^{(3/2)}(t')\,dt' \\ C_{n_0}^2 &= \sum_{n=2n_0+2}^{\infty} \frac{\Lambda(n)}{(n+3)^2} \end{aligned} \tag{A3}$$

We remark that

$$\tilde{M}^{-3/2}\frac{d\tilde{M}}{dt} = 2C_{n_0}[\exp(2\gamma_{n_0}t)]\tilde{N}^{3/2}\tilde{M}^{-3/2} \leqslant 2C_{n_0}\exp(-\gamma_{n_0}t)$$

Integrating both sides, we get

$$\tilde{M}^{-1/2}(t) > \tilde{N}^{-1/2}(0) - \frac{c_{n_0}}{\gamma_{n_0}} + \frac{c_{n_0}}{\gamma_{n_0}}\exp(-\gamma_{n_0}t)$$

If $\tilde{N}^{1/2}(0) < \gamma_{n_0}/c_{n_0}$, we can substitute into the rhs of (A3) and finally we find

$$\tilde{N}(t) \leqslant \frac{(\gamma_{n_0}/c_{n_0})^2\tilde{N}(0)}{\{\tilde{N}(0)^{1/2} + [\gamma_{n_0}/c_{n_0} - \tilde{N}(0)^{1/2}]\exp(\gamma_{n_0}t)\}^2} \tag{A4}$$

if

$$\tilde{N}(0) \leqslant \left(\frac{\gamma_{n_0}}{c_{n_0}}\right)^2 = \left(\frac{n_0+1}{n_0+3}\right)^2\left[\frac{2}{15}\sum_{n=2n_0+2}^{\infty}\frac{(2n+3)(2n+5)}{(n+1)(n+2)(n+3)^2}\right]^{-1}$$

A3. Bounds on $N_0(t) = \sum |a_n(t)|\lambda_n$. We start from Eq. (9c), multiply by λ_n, take the modulus of both sides, and bound the rhs,

$$|\lambda_n a_n(t)| = |\lambda_n a_n(0)|\exp(-\gamma_n t), \qquad n \leqslant 2n_0 + 1$$

$$|a_n(t)\lambda_n| \leqslant [\exp(-\gamma_n t)]\left[|a_n(0)\lambda_n| + (n+3)^{-1}\int_0^t \exp(\gamma_n t') \right. \tag{A5}$$

$$\left. \times \sum_{m+p=n-2} |\lambda_m a_m^{(t')}||\lambda_p a_p(t')|\frac{\lambda_n\,dt}{\lambda_n\lambda_p}\right], \qquad n \geqslant 2(n_0+1)$$

We obtain upper bounds on the rhs by the substitution $\gamma_n \to \gamma_{n_0}$, $\lambda_n/\lambda_m\lambda_p \to \Lambda(n)$, $\Lambda(n)/(n+3) \to (2n_0+5)^{-1}\Lambda(2n_0+2)$:

$$|\lambda_n a_n| \leqslant |\lambda_n a_n(0)|\exp(-\gamma_{n_0}t)$$

$$|\lambda_n a_n| \leqslant [\exp(-\gamma_{n_0}t)]\left[|a_n(0)\lambda_n| + \frac{\Lambda(2n_0+2)}{2n_0+5}\right.$$

$$\left. \times \int_0^t \exp(\gamma_{n_0}t')\sum |\lambda_m a_m||\lambda_p a_p|\,dt'\right]$$

Summing over n, we get a nonlinear integral inequality:

$$N_0(t)\exp(\gamma_{n_0}t) \leqslant M_0(t) = N_0(0) + \frac{\Lambda(2n_0+2)}{2n_0+5}\int_0^t N_0{}^2(t')\exp(\gamma_{n_0}t')\,dt' \tag{A6}$$

We note that

$$-\frac{d}{dt}M_0^{-1}(t) = (2n_0+5)^{-1}\Lambda(2n_0+2)M_0^{-2}[\exp(\gamma_{n_0}t)]N_0{}^2$$

$$< (2n_0+5)^{-1}\Lambda(2n_0+2)\exp(-\gamma_{n_0}t)$$

Integrating both sides, we get

$$N_0(0)[M_0(t)]^{-1} > [1 - N_0(0)\Lambda(2n_0 + 2)[\gamma_{n_0}(2n_0 + 5)]^{-1} + N_0(0)[\gamma_{n_0}(2n_0 + 5)]^{-1}\Lambda(2n_0 + 2)\exp(-\gamma_{n_0}t)]$$

if

$$N_0(0) < \gamma_{n_0}(2n_0 + 5)[\Lambda(2n_0 + 2)]^{-1}$$

We can substitute into the rhs of (A6) and obtain

$$N_0(t) \leqslant \frac{N_0(0)\gamma_{n_0}(2n_0 + 5)[\Lambda(2n_0 + 2)]^{-1}}{N_0(0) + [\gamma_{n_0}(2n_0 + 5)/\Lambda(2n_0 + 2) - N_0(0)]\exp(\gamma_{n_0}t)} \tag{A7}$$

if

$$N_0(0) \leqslant \frac{\gamma_{n_0}(2n_0 + 5)}{\Lambda(2n_0 + 2)} = \frac{15}{2}\frac{n_0 + 1}{n_0 + 3}\frac{(2n_0 + 3)(2n_0 + 4)(2n_0 + 5)}{(4n_0 + 7)(4n_0 + 9)}$$

A4. Bounds on $N_q(t) = \sum_{n_0} \lambda_n |a_n|(n + 2)(n + 1)\cdots(n + 3 - q)$, $q \geqslant 1$. We start with Eq. (9a), which we differentiate $q - 1$ times with respect to u. Equating to zero the coefficients of u^{n+3-q}, we get a nonlinear system like Eq. (9b) (where the nonlinear part appears for $n \geqslant 2n_0 + 2$), which we integrate from 0 to t [like Eq. (9c)]

$$\begin{aligned}(n + 2)\cdots(n + 3 - q)[a_n(t)] \\ &= (n + 2)\cdots(n + 4 - q)[(n + 3 - q)a_n(0) + qa_n(0)]e^{-t} \\ &\quad - q(n + 2)\cdots(n + 4 - q)a_n(t) \\ &\quad + e^{-t}\int_0^t dt'\, e^{t'}\Bigg[2(n + 2)\cdots(n + 4 - q)a_n(t') \\ &\quad + \sum_{p=0}^{q-1} C_{q-1}^p \sum_{m+m'=n-2} a_m(t')(m + 2)\cdots(m + 3 - p)a_{m'}(t') \\ &\quad \times (m' + 2)\cdots(m' + 3 - q + 1 + p)\Bigg]\end{aligned} \tag{A8}$$

with $(m + 2)\cdots(m + 3 - p) \equiv 1$ if $p = 0$. In (A8) the nonlinear part (last term on the rhs) appears for $n \geqslant 2n_0 + 2$.

We take the modulus of both sides, bound the rhs by the modulus of the different terms, multiply by λ_n, bound $\lambda_n/\lambda_m\lambda_{n-m-2}$ by $\Lambda(2n_0 + 2)$, sum over n, and find

$$\begin{aligned}N_q(t) &\leqslant e^{-t}[N_q(0) + qN_{q-1}(0)] + qN_{q-1}(t) \\ &\quad + e^{-t}\int_0^t e^{t'}\Bigg[2N_{q-1}(t') + \Lambda(2n_0 + 2) \\ &\quad \times \sum_{p=0}^{q-1} C_{q-1}^p N_p(t')N_{q-1-p}(t')\Bigg]dt'\end{aligned} \tag{A9}$$

We could use (A9) in order to get bounds on $N_q(t)$. However, we remark that $\Lambda(2n_0 + 2) < 1$, and substituting this value in the rhs of (A9), we get the same inequalities as in Ref. 3.

A5. Bounds on $\tilde{N}(t) = \sum_{-2} a_n^2(t)\lambda_n$. In Eq. (9b′) let us put $n + 2 = p$ and $b_p(t) = a_{p-2}(t)$; we obtain

$$\frac{db_p}{dt} + b_p = (p+1)^{-1} \sum_{m+q=p} b_m b_q, \qquad \tilde{N}(t) = \sum_0^\infty b_p{}^2(t)\bar{\lambda}_p \tag{A10}$$

where $\bar{\lambda}_p = \lambda_{p-2} = (2p+1)!!\, 2^{-p}(p!)^{-1}$. First we want to prove

$$\bar{\lambda}_p/\bar{\lambda}_m\bar{\lambda}_{p-m} \leqslant 1 \qquad \text{for } m \text{ integer} \in [0, p] \tag{A11}$$

We notice that $\bar{\lambda}_p/\bar{\lambda}_m\bar{\lambda}_{p-m} = 1$ for $m = 0$ or p; equals $(2p+1)/3p \leqslant 1$ for $m = 1$ or $p - 1$; and equals $\Lambda(p-2) < 1$ for $m \in [2, p-2]$. Second, we multiply (A10) by $\bar{\lambda}_p b_p$, integrate from 0 to t, and sum over p:

$$\tilde{N}(t)e^{2t} = \tilde{N}(0) + \int_0^t e^{2t'} \sum_p b_p \bar{\lambda}_p^{1/2}(p+1)^{-1} \sum_{m=0}^p b_m b_{p-m} \bar{\lambda}_p^{1/2}\, dt' \tag{A12}$$

Using (A.11) and the Schwarz inequality, we get

$$\left|\sum b_m b_{p-m}(\bar{\lambda}_p)^{1/2}\right| \leqslant \left(\sum b_m{}^2\bar{\lambda}_m \sum b_{p-m}^2 \bar{\lambda}_{p-m}\right)^{1/2} \leqslant \tilde{N}(t)$$

$$\left|\sum b_p \bar{\lambda}_p^{1/2}(p+1)^{-1}\right| \leqslant [\tilde{N}(t)]^{1/2}\left[\sum (p+1)^{-2}\right]^{1/2}$$

and finally

$$\tilde{N}(t)e^{2t} \leqslant \tilde{N}(0) + 2\pi\sqrt{6} \int_0^t e^{2t'} \tilde{N}^{3/2}(t')\, dt' \tag{A13}$$

This integral inequality was also considered and solved in Ref. 3:

$$\tilde{N}(t) \leqslant \tilde{N}(0)\{[1 - (\pi/\sqrt{6})\tilde{N}^{1/2}(0)]e^t + (\pi/\sqrt{6})\tilde{N}^{1/2}(0)\}^{-2} \qquad \text{if } \tilde{N}(0) \leqslant 6\pi^{-2} \tag{A14}$$

Third we multiply (A10) by $\bar{\lambda}_p^{1/2}$, integrate from 0 to t, sum over p, and define $M(t) = \sum (\bar{\lambda}_p)^{1/2}|b_p(t)|$; we get

$$M(t)e^t \leqslant M(0) + \sum_p \int_0^t e^{t'} \sum |b_m|\,|b_{p-m}|\bar{\lambda}_p^{1/2}\, dt'$$

Still using (A11), we finally get

$$M(t)e^t \leqslant M(0) + \int_0^t e^{t'} M^2(t')\, dt'$$

which is also an integral inequality solved in Ref. 3:

$$M(t) \leqslant M(0)\{[1 - M(0)]e^t + M(0)\}^{-1} \qquad \text{if } M(0) \leqslant 1$$

Noticing that $\tilde{N}(t) \leqslant M^2(t)$, we conclude that $\tilde{N}(t) < \infty$ if either $\tilde{N}(0) \leqslant 6\pi^{-2}$ or $M(0) \leqslant 1$.

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